

Solution for a non linear Schrodinger equation via Hopf-Cole transformation

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Abstract

In this short letter we show that the one dimensional non linear Schrödinger equation(NLS)can be solved by a Hopf-Cole transformation which converts it to the Burgers equation in turbulence.

1 General remarks

NLS often outbreaks in various fields of the applied physics.Many authors both mathematician and physicists investigated it's analytical properties[1].For a NLS, one can not obtains a compact solution even when we restrict ourselves only to a simple dimensional reduction of the independent variables to 2,i.e only x, t .Recently a NLS proposed which new non linear term is gravitational potential that arises only due to the entropic force acting on a particle with the aid of the entropy defined from quantum mechanics[2].

As a simple direct evidence of the seem of a NLS,suppose that a free particle with mass m moves in flat spacetime.We know that in some scenarios of the early universe, Planck constant spotted as an effective momentum dependence function ,via Generalized Uncertainty Principle (GUP)[3] and not constant regardless on some further developments in non commutative geometry. The idea that we introduce here can be staminate to the mass and regard it as an effective function dependence on the momentum of position[4].The simple presumption may be a linear dependence to the momentum in the \hbar or m .Everybody can treat with it as an effective Schrödinger equation for a test particle(free in rest) that a linear stimulus with unknown quantum essence disturb it's mass or further some GUP corrections have been replaced the usual Planck constant with a momentum dependence effective one. One applicable is the following NLS,

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\psi'' - i\hbar l^{\frac{1}{2}}c\psi'\psi \quad (1)$$

with $\psi(x, t), \dot{f} = \partial_t f, f' = \partial_x f$.It is salutary for reader if we alight dimensions of this new term. We take the light speed $c = 1$ (Geometrical units),and for accuracy of the usual dimension of the wave function in one dimensional problems we introduced a unit length scale $l = 1$ (can be regarded as the inverse square of the norm of the wave function i.e. $l^{-1} = \sqrt{\|\psi\|_{C^2}}$.We read (1) as the entropic efficacy of the Holographic screen of the test particle in the neighborhood screen in the new explanation of the Gravity as an entropic

force exerted on the particle near a holographic screen(equip potential surfaces) in an emergence model of the spacetimes [5].

1.1 Exact solution

Anyway the equation (1) is a simple NLSE, and by definition

$$\varepsilon = \frac{i\hbar}{2m} \quad (2)$$

converts to the following famous equation in non linear PDE, Burgers' eq[6],

$$\dot{\psi} = \varepsilon\psi'' - \psi\psi' \quad (3)$$

Burgers' equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. It is named for Johannes Martinus Burgers'. Now consider the planar 2 dimensional vector field,

$$V = (-u, 0.5\psi^2 - \varepsilon\psi') \quad (4)$$

Obviously is Curl free, i.e. a conservative vector field. Thus according to the familiar theorems, there exists a scalar potential function which obeys the next conditions,

$$\phi' = -\psi, \dot{\phi} = 0.5\psi^2 - \varepsilon\psi' \quad (5)$$

The new scalar function ϕ solves the equation

$$\dot{\phi} = 0.5\phi'^2 + \varepsilon\phi'' \quad (6)$$

Letting $\phi = 2\varepsilon \log(\eta)$, then we are left with

$$\dot{\eta} - \varepsilon\eta'' = 0 \quad (7)$$

The one dimensional Heat equation or free particle Schrödinger wave equation and from (5) we obtain

$$\psi = -2\varepsilon \frac{\eta'}{\eta} \quad (8)$$

Which is the Hopf-Cole transformation[7]. The Burgers' equation then been linearized by the Cole-Hopf transformation. Some solutions of the Burgers equation has been investigated by Majid and Ranasinghe [8]. The general solution for (3) with Initial Conditions (IC) $\psi_0(x) = \psi(x, 0)$ is

$$\psi(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \psi_0(y) K(x, y; t) dy}{\int_{-\infty}^{\infty} K(x, y; t) dy} \quad (9)$$

Where in it the heat kernel or propagator for a free particle in usual can be written as,

$$K(x, y; t) = \exp\left(\frac{im(x-y)^2}{2t\hbar}\right) \quad (10)$$

WE can related this propagator to the partition function of a statistical system by replacing the temperature with a Wick rotated time scale.

1.2 Localized particle

Now for a particle which is localized at origin at time $t = 0$ we can substitute the initial non renormalizable wave function

$$\psi_0(x) = N\delta(x) \quad (11)$$

We know that the wave function for a localized particle is not renormalizable. In some texts the authors set $N = 1$. But we exempt N from this, regard it as a free constant. and if we assume that for initial data we must have

$$\lim_{|x| \rightarrow \infty} \frac{1}{x^2} \int_a^x \psi_0(y) dy \rightarrow 0 \quad (12)$$

(a is arbitrary and never affect the value of the wave function, choosing $a=1$) and after simple integration we have

$$\psi(x, t) = \sqrt{\frac{2i\hbar}{m\pi t}} \frac{\exp(\frac{imx^2}{2\hbar t})}{2(\exp(-imN/\hbar) - 1)^{-1} + \frac{\sqrt{\pi}}{2}(1 - \operatorname{erf}(\frac{x\sqrt{m}}{\sqrt{2i\hbar t}}))} \quad (13)$$

where $\operatorname{erf}(x) \equiv \int_0^x e^{-z^2} dz$ is the integral of the Gaussian distribution. As [2] we suggest that the probability density $|\psi(x, t)|$ is in fact related to a partition function Z for different possible states, in the location of the statistical mechanics. Thus we derived the partition function Z for different possible quantum states in the NLS evolutionary scheme.

2 Summary

NLS equations arise in different theoretical and applied problems. One of the most class of the solvable models due to the Burgers'. In this work we look on the non linear terms in the Burgers' equation as the entropic corrections to the free particle Schrödinger equation in the frame of the new model proposed by Verlinde[5]. We show that under influence of this non linear term, the time evolution of a localized particle is so different from the common quantum mechanics. We related the norm of the wave function to the partition function of the system and also explicitly we calculate the wave function for it.

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